Exam Analysis on Manifolds

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January 30, 2018

This exam consists of **five** assignments. The first four allow for short solutions. You get 10 points for free.

Assignment 1. (9+6=15 pt.)
Let M be an n-dimensional C^∞-manifold (without boundary), with $n \geq 1$.

1. Assume that M is compact, and let $\varphi : M \to \mathbb{R}$ be a C^∞-function. Prove that there are at least two points at which the one-form $d\varphi$ is zero. (Hint: note that φ has a maximum and a minimum on M.)

2. Give an example of a non-compact manifold M for which the previous claim does not hold.

Assignment 2. (5+5+5=15 pt.)
Let ω be the one-form on $\mathbb{R}^3$ given by $\omega = x\,dy - y\,dx + z\,dz$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be the map given by $\varphi(u, v) = (\cos u, \sin u, v)$.

1. Compute the one-form $\varphi^*\omega$ on $\mathbb{R}^2$.

2. Prove that ω is not exact.

3. Prove that $\varphi^*\omega$ is exact, and determine a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that $\varphi^*\omega = d\psi$.

Assignment 3. (5+10=15 pt.)
Let M be an n-dimensional C^∞-manifold (without boundary), with $n \geq 1$.

1. Prove that every non-empty open subset of $\mathbb{R}^n$ is a C^∞-manifold.

2. Let (U, f) be a chart of a maximal C^∞-atlas on M. Prove that f is a C^∞-map.

Assignment 4 and 5 on next page

Assignment 4. (6+6+8=20 pt.)
Let $\omega \in \Lambda^2(\mathbb{R}^3)^*$ be non-zero.

1. Prove that there are two independent vectors $v_1, v_2 \in \mathbb{R}^3$ for which $\omega(v_1, v_2) = 1$.

2. Suppose that we can extend the system $\{v_1, v_2\}$ of Part 1 to a basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ such that $\omega(v_1, v_3) = 0$ and $\omega(v_2, v_3) = 0$. Prove that $\omega = v_1^\circ \wedge v_2^\circ$.
   (Recall that $\{v_1^\circ, v_2^\circ, v_3^\circ\}$ is the dual basis of a basis $\{v_1, v_2, v_3\}$.)

3. Prove that the system $\{v_1, v_2\}$ can be extended to a basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ such that $\omega = v_1^\circ \wedge v_2^\circ$.

Assignment 5. (5+5+5+5+5=25 pt.)
Let $M = \mathbb{R}^n \setminus \{O\}$, and let the differential form ω of degree $n-1$ on M be given by

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{(x_1^2 + \cdots + x_n^2)^{n/2}} \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Here $O = (0, \ldots, 0) \in \mathbb{R}^n$. The hat over a symbol means that this symbol is omitted. Furthermore, the differential form η of degree $n-1$ on M is defined by

$$\eta = \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

1. Prove that ω is closed

2. Let $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ be the unit $(n-1)$-sphere with center at O, and let $i : \mathbb{S}^{n-1} \to M$ be the inclusion map. Show that $\int_{\mathbb{S}^{n-1}} i^*\eta \neq 0$. (Hint: $\mathbb{S}^{n-1}$ is the boundary of the closed unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$; observe that η is defined on the full set $\mathbb{B}^n$.)

3. Conclude that $\int_{\mathbb{S}^{n-1}} i^*\omega \neq 0$. (Hint: prove that $i^*\omega = i^*\eta$.)

4. Prove that ω is not exact.

5. Show that there is no diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n \setminus \{O\}$.

Solutions

Assignment 1.

1. Since M is compact, φ attains its maximum value at some point $p \in M$. Let $v \in T_pM$, say $v = c'(0)$ for some differentiable curve $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$. Then $\varphi \circ c : (-\varepsilon, \varepsilon) \to \mathbb{R}$ has a maximum value at $0 \in \mathbb{R}$, so $d\varphi_p(v) = (\varphi \circ c)'(0) = 0$. Therefore, $d\varphi_p = 0$. Since φ also has a minimum, the one-form $d\varphi$ is also zero at the point at which the minimum is attained. (If the maximum and the minimum are equal, φ is constant, so $d\varphi$ is identically zero on M. Note that M is n-dimensional with $n \geq 1$, so it contains at least two points.)

2. Let $M = \mathbb{R}$, and let $\varphi(x) = x$. Then $d\varphi = dx$ is nowhere zero on M.

Assignment 2.

1. $\varphi^*\omega = \cos u \, d(\sin u) - \sin u \, d(\cos u) + v dv = du + v \, dv$.

2. $d\omega = 2dx \wedge dy$, which is a nonzero two-form on $\mathbb{R}^3$. So ω is not closed, and, therefore, not exact.

3. $d(\varphi^*\omega) = d(du) + dv \wedge dv = 0$, so $\varphi^*\omega$ is closed. By Poincaré's Lemma, it is also exact. In particular, $\varphi^*\omega = d\psi$ for $\psi(u, v) = u + \frac{1}{2}v^2$.

Assignment 3.

1. Let $\mathrm{id} : U \to U$ be the identity map. Then $\{(U, \mathrm{id})\}$ is an atlas defining a C^∞-structure on U. (Note that this atlas is not maximal.)

2. Let $\{(U_\alpha, f_\alpha)\}_{\alpha \in I}$ be a maximal atlas of M containing the chart (U, f), say $f = f_\beta$. To prove that f is C^∞ at an arbitrary point p of U, we have to prove that its expression in local coordinates around p in U and around $f(p)$ in M is a C^∞-map. So take $\alpha \in I$ such that $f(p) = f_\beta(p) \in f_\alpha(U_\alpha)$. We have to find a chart (V, g) on U such that $f_\beta(g(V)) \subset f_\alpha(U_\alpha)$ and the map $f_\alpha^{-1} \circ f_\beta \circ g : V \to U_\alpha$ is differentiable (as a map from the open subset V in $\mathbb{R}^n$ to the open subset U_α in $\mathbb{R}^n$). Let $V = f_\beta^{-1}(f_\beta(U_\beta) \cap f_\alpha(U_\alpha))$ and let $g : V \to U_\beta$ be the inclusion map, then g is a C^∞-map. Since $f_\alpha^{-1} \circ f_\beta$ is the transition map of the charts (U_α, f_α) and (U_β, f_β), it is a C^∞-map. Therefore, the composition $f_\alpha^{-1} \circ f_\beta \circ g$ is a C^∞-map.

Remark. We gave partial credits if you based the proof on the observation that $f_\alpha^{-1} \circ f$ is a transition map between charts of a C^∞-atlas on M, but forgot to use an atlas on U.

Assignment 4.

1. Since ω is non-zero, there are vectors $w_1, w_2 \in \mathbb{R}^3$ such that $\omega(w_1, w_2) \neq 0$. The latter inequality implies that $\{w_1, w_2\}$ is an independent system. The vectors v_1 and v_2, defined by $v_1 = \omega(w_1, w_2)^{-1} w_1$ and $v_2 = w_2$, are independent and satisfy $\omega(v_1, v_2) = 1$.

2. The claim follows from the following decomposition of ω:

$$\omega = \omega(v_1, v_2) v_1^\circ \wedge v_2^\circ + \omega(v_1, v_3) v_1^\circ \wedge v_3^\circ + \omega(v_2, v_3) v_2^\circ \wedge v_3^\circ.$$

3. So we have to find v_3 such that $\omega(v_1, v_3) = 0$ and $\omega(v_2, v_3) = 0$. First extend the system $\{v_1, v_2\}$ to a basis $\{v_1, v_2, w_3\}$ of $\mathbb{R}^3$. Now take $v_3 = w_3 + a v_1 + b v_2$, such that $\omega(v_1, v_3) = 0$ and $\omega(v_2, v_3) = 0$. Since $\omega(v_1, v_2) = 1$, we have $a = \omega(v_2, w_3)$ and $b = -\omega(v_1, w_3)$.

Assignment 5.

1. This follows from a straightforward computation.

2. Let $j : \mathbb{S}^{n-1} \to \mathbb{B}^n$ be the inclusion map. Note that $j^*\eta = i^*\eta$. Now use Stokes:

$$\int_{\mathbb{S}^{n-1}} i^*\eta = \int_{\mathbb{S}^{n-1}} j^*\eta = \int_{\partial\mathbb{B}^n} j^*\eta = \int_{\mathbb{B}^n} d\eta = \int_{\mathbb{B}^n} n \, dx_1 \wedge \cdots \wedge dx_n = n \ \mathrm{Vol}(\mathbb{B}^n) \neq 0.$$

3. Let $f(x_1, \ldots, x_n) = (x_1^2 + \cdots + x_n^2)^{n/2}$. Then $i \circ f(x_1, \ldots, x_n) = 1$, so the claim in the hint follows. Therefore,

$$\int_{\mathbb{S}^{n-1}} i^*\omega = \int_{\mathbb{S}^{n-1}} i^*\eta \neq 0.$$

4. Suppose $\omega = d\sigma$ for a differential form of degree $n - 2$ on M. Then

$$\int_{\mathbb{S}^{n-1}} i^*\omega = \int_{\mathbb{S}^{n-1}} i^*(d\sigma) = \int_{\mathbb{S}^{n-1}} d(i^*\sigma) = \int_{\partial\mathbb{S}^{n-1}} i^*\sigma = 0,$$

since $\partial\mathbb{S}^{n-1} = \emptyset$. This contradiction shows that ω is not exact.

5. Suppose such a diffeomorphism exists. Then the pull-back $\phi^*\omega$ is closed, since $d(\phi^*\omega) = \phi^*(d\omega) = 0$. According to Poincaré's Lemma, there is a differential form τ of degree $n - 2$ on $\mathbb{R}^n$ such that $\phi^*\omega = d\tau$. Since ϕ is a diffeomorphism, we have $\omega = (\phi^{-1})^*(d\tau) = d((\phi^{-1})^*\tau)$, so ω is exact. In view of Part 4, this is a contradiction, so there is no such diffeomorphism.